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APPLICATION OF THE VON KÁRMÁN MOMENTUM THEOREM TO TURBULENT BOUNDARY LAYERS

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SUMMARY

A study is made of the Von Kármán momentum theorem with respect to its application to turbulent boundary layers in a positive pressure gradient. Although the Von Kármán momentum theorem for turbulent boundary layers contains momentum terms due to the fluctuating motion as well as momentum terms due to the mean motion, the general practice has been to neglect the momentum terms due to the fluctuating motion. Data were obtained from Schubauer and Klebanoff (NACA TN 2133) and Ludwieg and Tillmann (NACA TM 1285) with which both the terms due to the mean flow and the terms due to the fluctuating flow could be evaluated. The results indicate that the streamwise derivative of the turbulent longitudinal momentum $\rho\overline{u'u'}$ may be large near separation and therefore should be considered when the Von Kármán momentum theorem is used for turbulent boundary layers near separation.

INTRODUCTION

Because the Von Kármán momentum theorem (reference 1) is a relation between the changes in boundary-layer momentum and the external stresses of pressure gradient and wall shear, it applies equally well for turbulent boundary layers as for laminar boundary layers. For turbulent boundary layers, however, problems arise in the interpretation of momentum and in the application of the theorem to experimental data.

In a turbulent flow, the net motion is a combination of molecular motion, turbulent motion, and stream motion. Reynolds (reference 2) showed that the periods of oscillation associated with these three phases of motion are distinguishable by means of appropriate time averages. In aerodynamics, the primary interest in the molecular motion lies in its integrated effects, that is, pressure, temperature, and viscosity. In the study of turbulent motion, the fluid is generally considered to be a continuum and the net velocity at any point is assumed to be resolved into a fluctuating component and a mean component which is independent of time.

If the velocities appeared in the equations of motion only to the first power, a long-time average of the fluctuating component would be zero and the equations would contain terms involving only the mean velocities. Because, however, the momentum involves the squares of the velocities, momentum terms involving time averages of the squares and products of fluctuating components must be considered inasmuch as they are not zero. The expression for the Von Kármán momentum theorem as applied to turbulent boundary layers therefore contains momentum terms for the mean flow and momentum terms involving the fluctuating components.

Methods (references 3 to 5) for calculating the development of a turbulent boundary layer have, in general, used an empirical equation for wall shearing stress, a known pressure distribution, an empirical equation relating the shape of the velocity distribution to the external forces, and the Von Kármán momentum theorem to give the changes in boundary-layer momentum. The problem is determinate if the changes in boundary-layer momentum are assumed to be ascribed to the mean velocity alone. Calculations made with these assumptions have given good results in many instances but recently discrepancies have been noted, especially in the region of separation.

Discrepancies have also been noted when the Von Kármán momentum theorem was used to obtain wall shearing stress (references 6 to 9). By assuming that the changes in boundary-layer momentum could be completely described by the mean flow, measurements were made of the mean velocity and pressure gradient, and the Von Kármán momentum theorem was used as a balance to give the wall shearing stress. This method gave values of wall shearing stress that increased in a positive pressure gradient which is in contradiction to the data of references 10 to 12 which indicate that the wall shearing stress decreases in a positive pressure gradient.

The neglect in the Von Kármán momentum theorem, as it is usually applied to turbulent boundary layers, of the momentum terms involving the averages of the fluctuating velocities is a possible explanation of the observed discrepancies. This possibility is examined in the present paper. With the use of the Navier-Stokes equations as a starting point, an integral relation that includes terms resulting from the fluctuating motion as well as those associated with the mean flow is derived for the turbulent boundary layer. With the use of this relation and the experimental data given by Dryden (reference 10) and Schubauer and Klebanoff (reference 11), together with measurements of the wall shearing stress made independently by Ludwieg and Tillmann (reference 12), an attempt is made to evaluate the relative order of magnitude of the terms resulting from the fluctuating motion as compared with those resulting from the mean motion.

After the analysis presented herein had been completed, research reported by Wallis (reference 13) in Australia was made available. Wallis surveyed the equations of energy for turbulent boundary layers and suggested that "where the turbulent boundary layer flow is being rapidly accelerated or decelerated, the von Karman momentum equation will be inaccurate as it does not take account of the internal process by which the energy of mean motion is altered independently of an external force such as skin friction." Reference is made by Wallis to a private communication from B. G. Newman in which the problem is approached from momentum rather than energy considerations. Newman suggests that the Reynolds normal stresses modify the Von Kármán momentum theorem near separation. Apparently, data were not available with which a suitable check of the hypothesis could be made.

An analysis similar to that presented herein but much less detailed is included in a recent paper by Rubert and Persh (reference 14).

SYMBOLS

H	boundary-layer-velocity-profile shape parameter (δ^*/θ)
p	absolute static pressure at any point
t	time
u	streamwise velocity at a point in boundary layer
U	streamwise velocity at outer limit of boundary layer
v	normal velocity at a point in boundary layer
x	streamwise coordinate
y	normal coordinate
δ	boundary-layer thickness
δ^*	displacement thickness
θ	momentum thickness
μ	coefficient of viscosity
ν	coefficient of kinematic viscosity (μ/ρ)

ρ fluid density

τ viscous shearing stress

Subscripts:

m reference condition

0 wall boundary

δ outer limit of boundary layer

Primes denote fluctuating quantities and bars denote time averages.

ANALYSIS

The Navier-Stokes equations for a two-dimensional flow in which the viscosity and density are assumed constant and the body forces are negligible are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2}$$

The associated equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

By resolving the instantaneous values into mean and fluctuating components, by considering the mean flow steady and taking a long-time average of the fluctuating components, by considering the Prandtl boundary-layer assumptions valid for the mean flow up to separation, and by using the continuity equation, equations (1) and (2) may be written as

$$\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) + \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y}\right) = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + \nu\frac{\partial^2 \bar{u}}{\partial y^2} \tag{4}$$

$$\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} = -\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial y} \tag{5}$$

In the immediate neighborhood of separation, the validity of the boundary-layer assumptions may be open to question. The extent of the error introduced by making the boundary-layer assumptions under such circumstances is, however, not known. In any case, by retaining the terms due to turbulence, it is not implied that these terms are necessarily larger than those that have been neglected but that the magnitude of the effect of the terms due to turbulence is to be investigated. By combining equations (4) and (5) and by integrating through the boundary layer, the following equation of momentum is obtained (see appendix):

$$\frac{\tau_0}{\overline{\rho} U^2} - \frac{d\theta}{dx} - (H + 2)\frac{\theta}{U}\frac{dU}{dx} = -\frac{1}{U^2}\int_0^\delta \frac{\partial \overline{u'u'}}{\partial x}\,dy + \frac{1}{U^2}\int_0^\delta \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right)dy\,dy - \frac{\delta}{U^2}\int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2}\,dy \tag{6}$$

The Von Kármán momentum theorem for laminar boundary layers is

$$\frac{\tau_0}{\rho U^2} - \frac{d\theta}{dx} - (H + 2)\frac{\theta}{U}\frac{dU}{dx} = 0$$

The usual practice has been to consider this equation equally valid for turbulent flow.

Whether the Von Kármán momentum theorem can be used for turbulent boundary layers by considering only the momentums of the mean flow can be ascertained by experimental evaluation of the momentum terms on the right side of equation (6).

PRESENTATION OF DATA AND EVALUATION OF THE MOMENTUM TERMS

Data are presented from which the terms in the Von Kármán momentum theorem can be evaluated and the order of magnitude of the terms due to

turbulence can be compared with the magnitude of the terms due to the mean flow.

Mean-Flow Data

Presentation of data and discussion of accuracy.- The mean-flow data of Schubauer and Klebanoff (reference 11) have been plotted as $(U/U_m)^2$, θ, and H as functions of x in figures 1, 2, and 3, respectively. The original velocity data were studied in some detail, and the scatter of data is believed to be a good indication of the random errors in the measurements and in the integration procedure.

A systematic error occurs in the boundary-layer data because, in a turbulent air stream, pitot-tube measurements give values of velocity which are in excess of the mean local velocity. This error has been discussed by Goldstein in reference 15 and is shown to be

$$\sqrt{\bar{u}^2 + \overline{u'u'}} - \sqrt{\bar{u}^2}$$

The velocity data given by Schubauer and Klebanoff have not been corrected for the turbulence but sufficient data are presented to permit this computation. The magnitude of this correction to the velocity is illustrated in figure 4 for a point just before separation ($x = 25.4$ ft). The difference between the curves is somewhat greater than differences that might be indicated by random errors alone.

The velocity profiles have been corrected and integrated and the corrected values of θ are plotted in figure 2. At small values of x, that is, less than 21.5 feet, the correction to the velocity profile was negligible. Because the correction resulted in a decrease of velocity, the displacement thickness was increased and the values of corrected H were considerably higher. The corrected values of H are plotted in figure 3 and refaired with a dashed line. The corrected values of these parameters have been used in the subsequent computations.

Comparison of wall shearing stress as obtained by different methods.- The values of wall shearing stress estimated by Schubauer and Klebanoff are compared in figure 5 with the values of wall shearing stress obtained by use of the formulas of Squire and Young (reference 16), Falkner (reference 17), and Ludwieg and Tillman (reference 12). The flat-plate (zero pressure gradient) formula of Squire and Young has been used extensively in the United States in conjunction with the method of Von Doenhoff and Tetervin (reference 4) and the flat-plate formula of Falkner has been used in England in conjunction with the method of Garner (reference 5). The formula of Ludwieg and Tillman was the result

of some experiments made to determine wall shearing stress in a pressure gradient with the aid of a calibrated heat-transfer instrument.

Although the estimated values of wall shearing stress given by Schubauer and Klebanoff became zero at separation, at lower values of x these values of wall shearing stress are considerably higher than those given by the flat-plate formulas. A positive pressure gradient tends to decrease the velocity in the boundary layer and hence to decrease the normal gradient of velocity at the wall. Because the flow in the immediate neighborhood of the wall is laminar, the decrease in normal velocity gradient at the wall is directly proportional to the decrease in wall shearing stress. For all values of x larger than 17.5, the pressure gradient is positive; so the wall shearing stress would be expected to be less than that obtained by flat-plate formulas. Schubauer and Klebanoff (reference 11) estimated these values of wall shearing stress by fairing the $\rho\overline{u'v'}$ data to the region of the wall and then extrapolating the wall shearing stress by continuing the fairing to the wall with a slope equal to the pressure gradient. Although the trend of the data obtained by this method is correct, inaccuracies in the extrapolation resulted in absolute values that are apparently too large.

The distribution of wall shearing stress obtained by using the formula of Ludwieg and Tillmann appears to be the most reasonable from consideration of the experimental method used and from comparisons with well-established results for the zero-pressure-gradient case. Inspection of the Ludwieg and Tillmann formula

$$\frac{\tau_0}{\bar{\rho}U^2} = \frac{0.123}{10^{0.678H}} \frac{1}{R_\theta{}^{0.268}}$$

(where R_θ is the Reynolds number based on θ) reveals that H must be infinite for the wall shearing stress to disappear and an infinite H is, of course, physically unreasonable. At separation, however, the indicated value of wall shearing stress is small enough to be considered negligible.

Substitution of mean-flow data in the Von Kármán momentum theorem.- Values of $-d\theta/dx$ and $(H + 2)\frac{\theta}{U}\frac{dU}{dx}$ were obtained by use of figures 1, 2, and 3 and are plotted as a function of x in figure 6. The wall shearing stress $\tau_0/\bar{\rho}U^2$ computed from the formula of Ludwieg and Tillmann is also plotted in figure 6. If in the application of the Von Kármán momentum theorem to turbulent boundary layers only mean-flow quantities need be considered, the sum of the terms would be zero. The sum of the mean-flow terms is plotted as a function of x in figure 7, where it

is shown that for a large range of x consideration of mean-flow quantities alone gives good results; near separation, however, the sum has a large value. Good results are obtained up to a value of H of approximately 1.5.

Because changes in the fairing of θ will alter the value of $\frac{d\theta}{dx}$, it is of interest to know how much different the values of θ would have to be to make the sum of the mean-flow terms zero. The $\frac{d\theta}{dx}$ required to make the sum of the mean-flow terms zero was integrated with respect to x from 17.5 feet to 25.4 feet and the results are plotted as a dashed line in figure 2. The differences between the experimental curve and the required curve could not be attributed to differences in fairing.

Fluctuating-Flow Data

Presentation of data.- Values of $\frac{\sqrt{\overline{u'u'}}}{U_m}$ reported by Schubauer and Klebanoff have been plotted as a function of x and y in figure 8. The data were plotted "three dimensionally" to facilitate fairing. Successive planes of $x = \text{Constant}$ were indicated along a 45° axis at distances equal to particular values of x. By this method of plotting, curves of $y = \text{Constant}$ can be used to check the fairing and to aid the visualization of the surface. This method of plotting and fairing can be used because the function indicated by the fluctuating-flow data is regular up to separation.

Evaluation of the momentum terms due to turbulence.- The faired data of figure 8 and figure 1 were used to obtain the values necessary to evaluate the term

$$\frac{1}{U^2}\int_0^{\delta} \frac{\partial \overline{u'u'}}{\partial x}\, dy$$

which is plotted as a function of x in figure 7. The magnitude of this term is affected only moderately by the fairing of the data. The value of this function is small at lower values of x and becomes large only in the region of separation. A comparison of the value of this function with the sum of the mean-flow terms obtained from figure 6 indicates that, at least qualitatively, the term $\frac{1}{U^2}\int_0^{\delta} \frac{\partial \overline{u'u'}}{\partial x}\, dy$, together with the mean-flow terms may be sufficient to describe the momentum balance in the turbulent boundary layer up to separation. A more precise

discussion of the importance and absolute magnitude of this term can be given only after more data are made available.

In order to evaluate the term

$$\frac{1}{U^2}\int_0^\delta \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy\ dy$$

the data of Schubauer and Klebanoff were investigated. The magnitude of this term is affected by the differentiation and integration processes and for that reason is influenced by small changes in fairing. A calculation was made in which the integrand was believed to be large. The calculation indicated that the term

$$\frac{1}{U^2}\int_0^\delta \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy\ dy$$

was a small fraction of the term

$$\frac{1}{U^2}\int_0^\delta \frac{\partial \overline{u'u'}}{\partial x} dy$$

but the results were too inaccurate to warrant presentation.

In evaluating the term

$$\frac{\delta}{U^2}\int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2} dy$$

the data were sufficient to give a magnitude to $\frac{\partial \overline{u'v'}}{\partial x}$ but were insufficient to permit calculation of $\frac{\partial^2 \overline{u'v'}}{\partial x^2}$. Because, however, the factor δ appears in the numerator, it is believed that the magnitude of the term is small.

CONCLUDING REMARKS

Experimental data show that the streamwise derivative of the turbulent longitudinal momentum $\rho\overline{u'u'}$ may be of sufficient magnitude to require its inclusion in the application of the Von Kármán momentum theorem to turbulent boundary layers near separation. The data show that for the particular case considered the Von Kármán momentum theorem with only mean-flow terms gave satisfactory results up to a value of the boundary-layer-velocity-profile shape parameter H of approximately 1.5. A quantitative discussion of the importance of the fluctuating components can be made only when more data become available.

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APPENDIX

DERIVATION OF THE VON KÁRMÁN MOMENTUM THEOREM FOR TURBULENT BOUNDARY LAYERS

The simplified Navier-Stokes equations, as given previously by equations (4) and (5), are

$$\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) + \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y}\right) = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + \nu\frac{\partial^2 \bar{u}}{\partial y^2}$$

$$\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial y}$$

From the last of these two equations

$$\bar{p}_0 - \bar{p}_\delta = \bar{\rho}\int_0^\delta \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy = \bar{\rho}\int_0^\delta \frac{\partial \overline{u'v'}}{\partial x}\, dy$$

because for simplicity $\overline{v'v'}$ is assumed to approach zero at $y = \delta$ and at $y = 0$. Also

$$\bar{p}_0 - \bar{p} = \bar{\rho}\int_0^y \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy$$

Then

$$\bar{p} = \bar{p}_\delta + \bar{\rho}\int_0^\delta \frac{\partial \overline{u'v'}}{\partial x}\, dy - \bar{\rho}\int_0^y \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy$$

and

$$\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} = -U\frac{\partial U}{\partial x} + \frac{\partial}{\partial x}\int_0^\delta \frac{\partial \overline{u'v'}}{\partial x}\, dy - \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy \qquad \text{(A1)}$$

because

$$\frac{1}{\bar{\rho}} \frac{\partial \bar{p}_\delta}{\partial x} = - U \frac{\partial U}{\partial x}$$

By substituting equation (A1) into the first of the simplified Navier-Stokes equations, the following equation can be written for a turbulent boundary layer in a two-dimensional incompressible flow:

$$\left(\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y}\right) + \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y}\right) = U \frac{\partial U}{\partial x} - \frac{\partial}{\partial x} \int_0^\delta \frac{\partial \overline{u'v'}}{\partial x} dy +$$

$$\int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy + \nu \frac{\partial^2 \bar{u}}{\partial y^2} \tag{A2}$$

In order to describe the action of the turbulent boundary layer as a whole, equation (A2) is integrated with respect to y through the boundary layer:

$$\int_0^\delta \left(\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y}\right) dy + \int_0^\delta \left(\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y}\right) dy =$$

$$\int_0^\delta \left(U \frac{\partial U}{\partial x}\right) dy - \int_0^\delta \left(\frac{\partial}{\partial x} \int_0^\delta \frac{\partial \overline{u'v'}}{\partial x} dy\right) dy +$$

$$\int_0^\delta \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy\, dy + \int_0^\delta \nu \frac{\partial^2 \bar{u}}{\partial y^2} dy \tag{A3}$$

Now

$$\int_0^\delta \bar{u} \frac{\partial \bar{u}}{\partial x} dy = \frac{1}{2} \frac{\partial}{\partial x} \int_0^\delta \bar{u}^2 dy - \frac{1}{2} U^2 \frac{\partial \delta}{\partial x}$$

$$\int_0^\delta \nu \frac{\partial^2 \bar{u}}{\partial y^2} dy = \nu \left[\left(\frac{\partial \bar{u}}{\partial y}\right)_\delta - \left(\frac{\partial \bar{u}}{\partial y}\right)_0\right] = - \nu \left(\frac{\partial \bar{u}}{\partial y}\right)_0 = - \frac{\tau_0}{\bar{\rho}}$$

$$\int_0^\delta \left(\frac{\partial}{\partial x} \int_0^\delta \frac{\partial \overline{u'v'}}{\partial x} \, dy \right) dy = \delta \int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2} \, dy$$

and from continuity

$$\bar{v} = - \int_0^y \frac{\partial \bar{u}}{\partial x} \, dy$$

If these quantities are substituted in equation (A3), the following relation is obtained:

$$\frac{1}{2} \frac{\partial}{\partial x} \int_0^\delta \bar{u}^2 dy - \int_0^\delta \left(\int_0^y \frac{\partial \bar{u}}{\partial x} \, dy \right) \frac{\partial \bar{u}}{\partial y} \, dy = \frac{1}{2} \frac{\partial}{\partial x} \int_0^\delta U^2 dy -$$

$$\frac{\tau_0}{\bar{\rho}} - \int_0^\delta \frac{\partial \overline{u'u'}}{\partial x} \, dy - \delta \int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2} \, dy +$$

$$\int_0^\delta \int_0^y \frac{\partial}{\partial x} \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} \right) dy \, dy \tag{A4}$$

where

$$\int_0^\delta \frac{\partial \overline{u'v'}}{\partial y} \, dy = (\overline{u'v'})_\delta - (\overline{u'v'})_0 = 0$$

Integration by parts gives

$$\int_0^\delta \left(\int_0^y \frac{\partial \bar{u}}{\partial x} \, dy \right) \frac{\partial \bar{u}}{\partial y} \, dy = U \frac{\partial}{\partial x} \int_0^\delta \bar{u} \, dy - \frac{1}{2} \frac{\partial}{\partial x} \int_0^\delta \bar{u}^2 dy - \frac{U^2}{2} \frac{\partial \delta}{\partial x} \tag{A5}$$

Substitute equation (A5) in equation (A4) and divide by U^2; then

$$\frac{1}{U^2}\frac{\partial}{\partial x}\int_0^\delta \bar{u}^2 dy - \frac{1}{U}\frac{\partial}{\partial x}\int_0^\delta \bar{u}\ dy + \frac{1}{2}\frac{\partial \delta}{\partial x} = \frac{1}{2}\frac{1}{U^2}\frac{\partial}{\partial x}\int_0^\delta U^2 dy -$$

$$\frac{\tau_0}{\bar{\rho}U^2} - \frac{1}{U^2}\int_0^\delta \frac{\partial \overline{u'u'}}{\partial x} dy - \frac{\delta}{U^2}\int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2} dy +$$

$$\frac{1}{U^2}\int_0^\delta \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy\ dy \qquad \text{(A6)}$$

By adding and subtracting

$$\frac{1}{2}\frac{1}{U^2}\frac{\partial}{\partial x}\int_0^\delta U^2 dy = \frac{1}{U}\frac{\partial}{\partial x}\int_0^\delta U\ dy - \frac{1}{2}\frac{\partial \delta}{\partial x}$$

the following equation can be obtained

$$\frac{1}{U^2}\frac{\partial}{\partial x}\int_0^\delta \left(\bar{u}^2 - U^2\right) dy - \frac{1}{U}\frac{\partial}{\partial x}\int_0^\delta (\bar{u} - U)\ dy + \frac{\tau_0}{\bar{\rho}U^2} =$$

$$-\frac{1}{U^2}\int_0^\delta \frac{\partial \overline{u'u'}}{\partial x} dy - \frac{\delta}{U^2}\int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2} dy + \frac{1}{U^2}\int_0^\delta \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy\ dy$$

(A7)

Let the momentum loss M of the stream be defined by

$$M = \bar{\rho}U^2\theta$$

where θ is the momentum thickness, then

$$\bar{\rho}U^2\theta = \bar{\rho}\int_0^\delta \left(\bar{u}U - \bar{u}^2\right) dy$$

or

$$\theta = \int_0^\delta \frac{\overline{u}}{U}\left(1 - \frac{\overline{u}}{U}\right) dy = \theta(x)$$

Now

$$\frac{d\theta}{dx} = -\frac{1}{U^2}\frac{d}{dx}\int_0^\delta \left(\overline{u}^2 - U^2\right) dy + \frac{1}{U}\frac{d}{dx}\int_0^\delta (\overline{u} - U)\ dy - \frac{1}{U}\frac{dU}{dx}(2\theta + \delta^*) \tag{A8}$$

where δ^*, the displacement thickness, is defined by

$$\delta^* = \int_0^\delta \left(1 - \frac{\overline{u}}{U}\right) dy = \delta^*(x)$$

By using equation (A8), equation (A7) may be written finally as

$$\frac{\tau_0}{\rho U^2} - \frac{d\theta}{dx} - (H + 2)\frac{\theta}{U}\frac{dU}{dx} = -\frac{1}{U^2}\int_0^\delta \frac{\partial \overline{u'u'}}{\partial x} dy +$$
$$\frac{1}{U^2}\int_0^\delta \int_0^y \frac{\partial}{\partial x}\left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}\right) dy\ dy - \frac{\delta}{U^2}\int_0^\delta \frac{\partial^2 \overline{u'v'}}{\partial x^2} dy$$

where H is defined as δ^*/θ. This equation is the momentum theorem for turbulent boundary layers and is given as equation (6) in the body of the paper.

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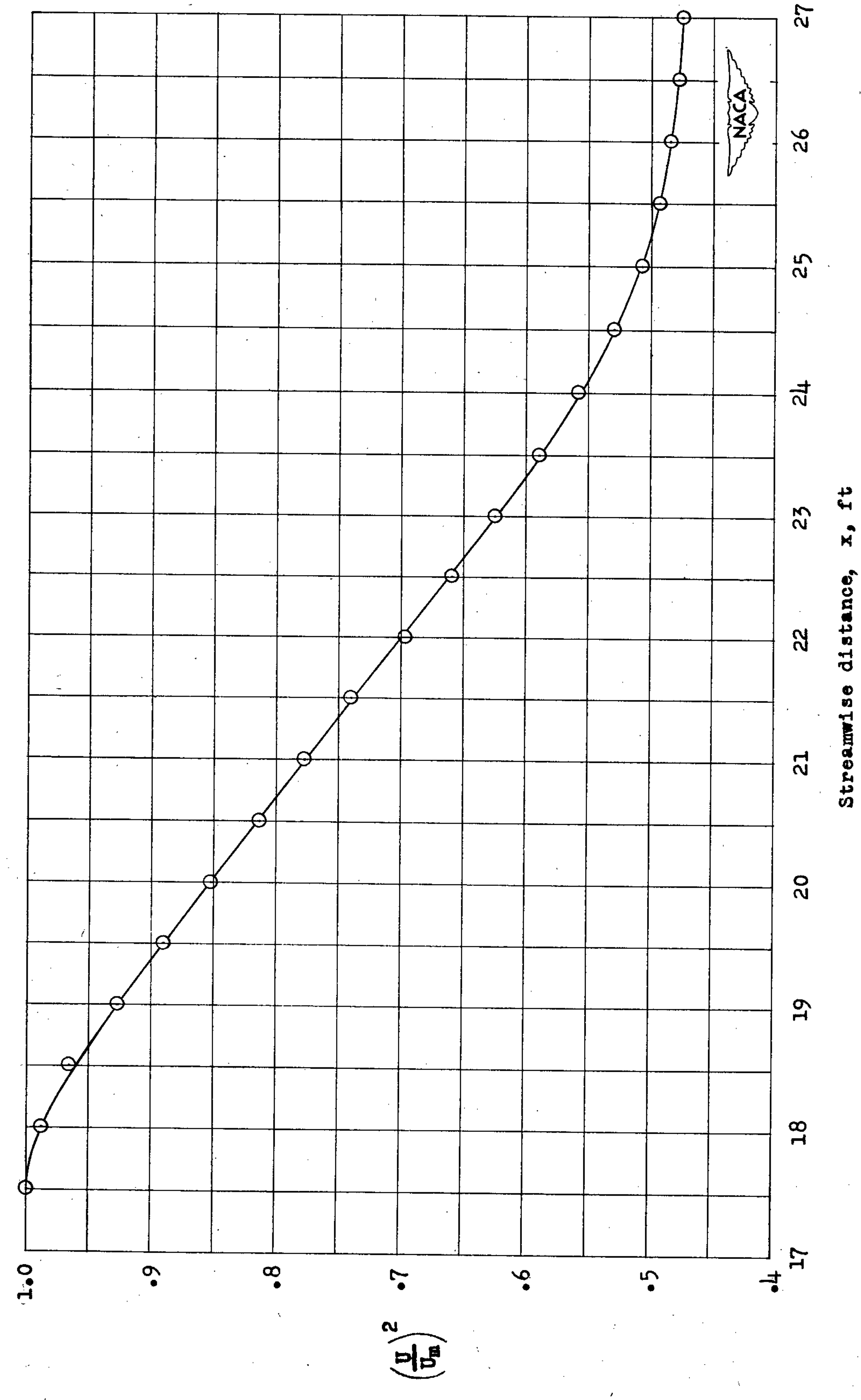

Figure 1.- Streamwise variation of the square of the ratio of the velocity at the outer limit of the boundary layer to a reference velocity. Data were obtained from reference 11.

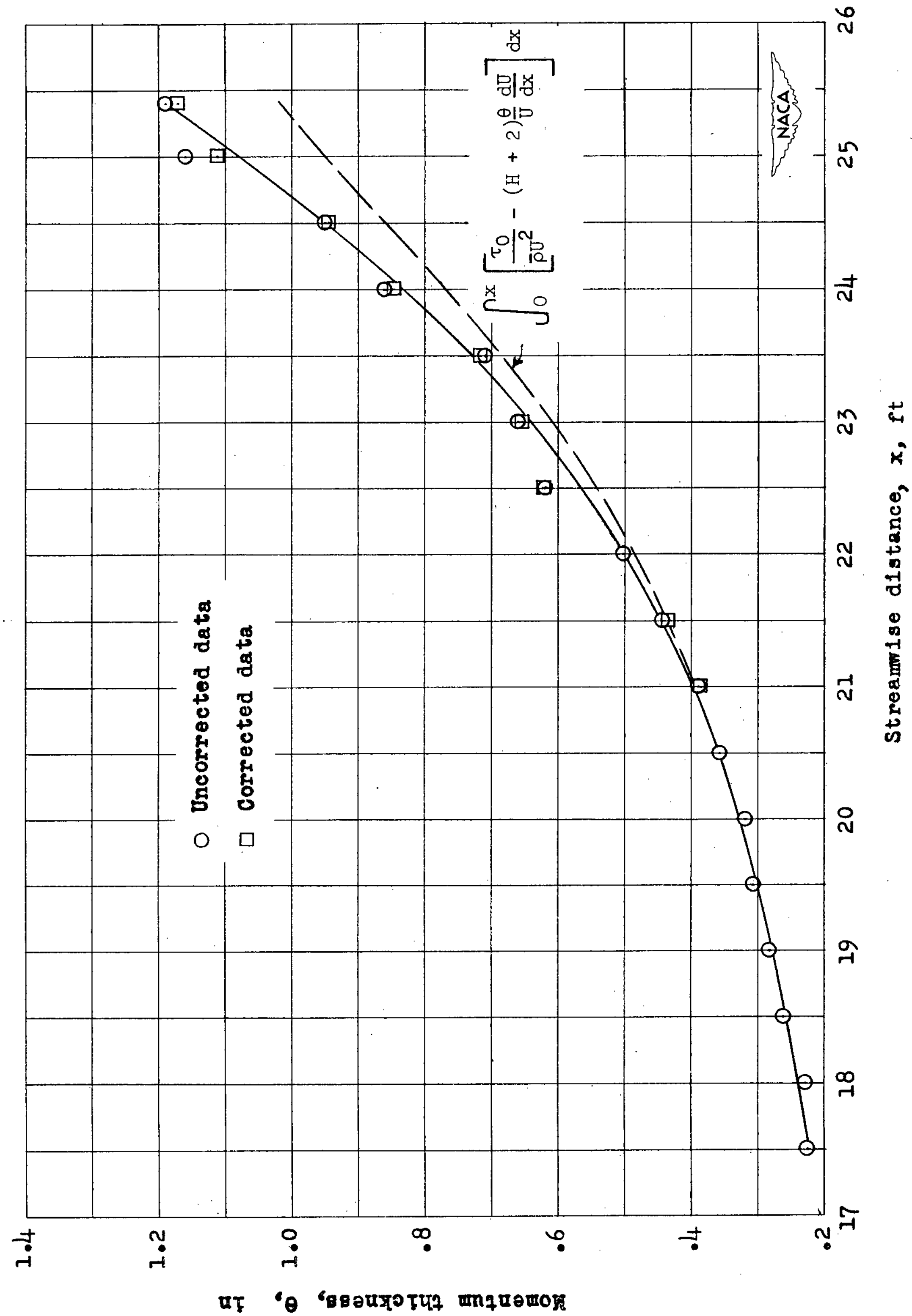

Figure 2.- Streamwise variation of the boundary-layer momentum thickness. Data were obtained from reference 11.

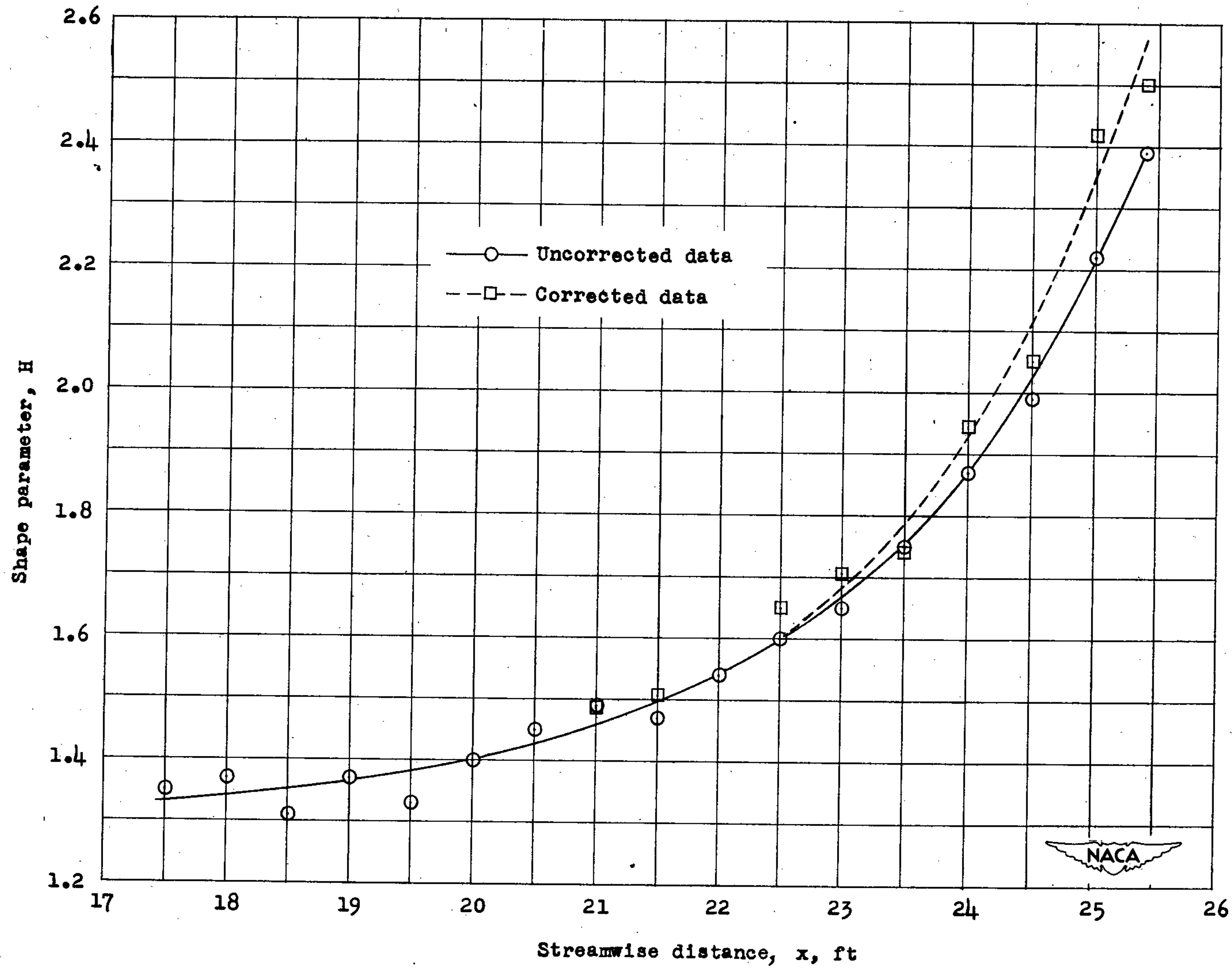

Figure 3.- Streamwise variation of the boundary-layer shape parameter H. Data were obtained from reference 11.

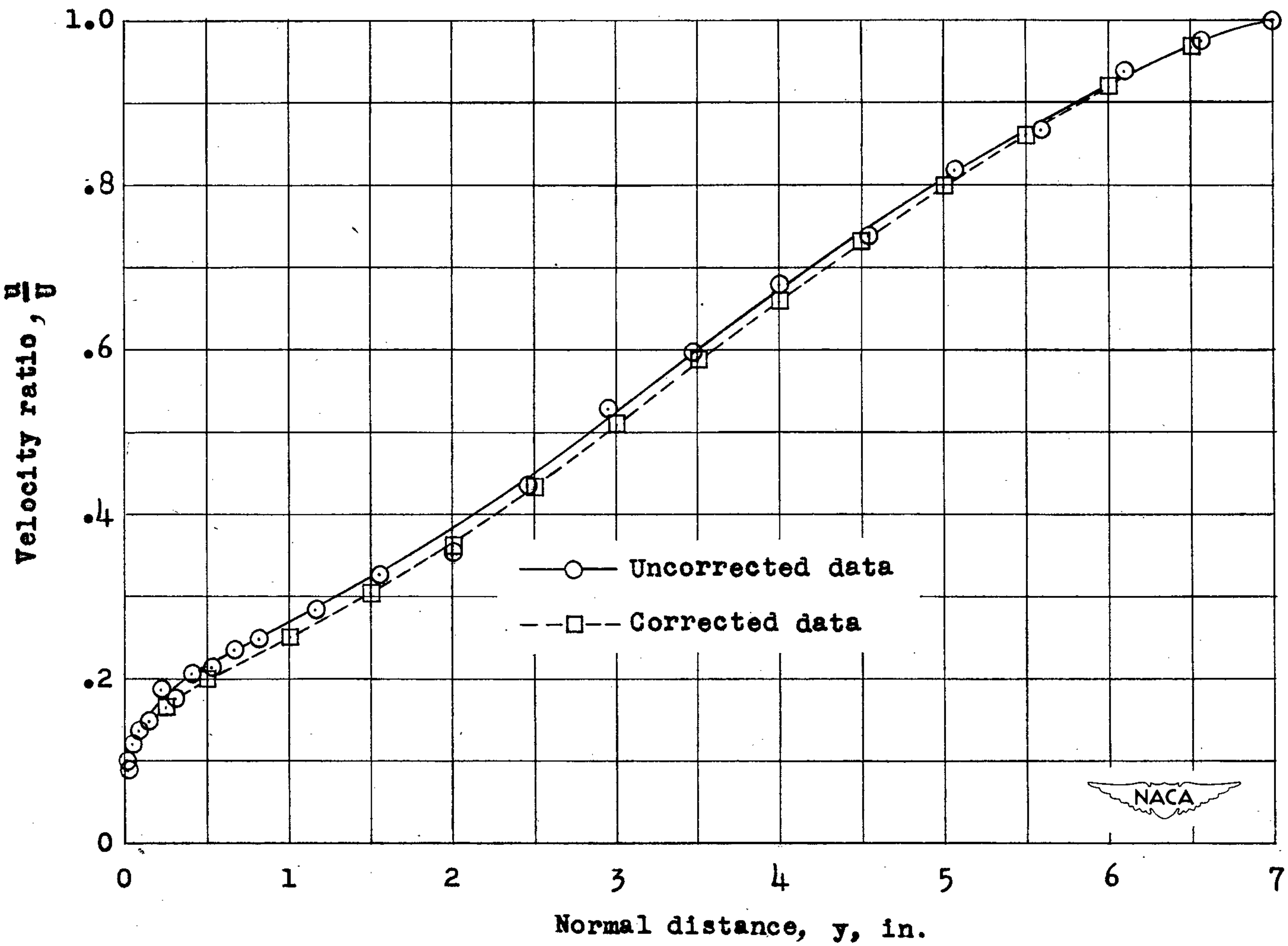

Figure 4.- Effect on boundary-layer-velocity distribution of the correction factor to the pitot-tube measurements. Data were obtained from reference 11. $x = 25.4$ feet.

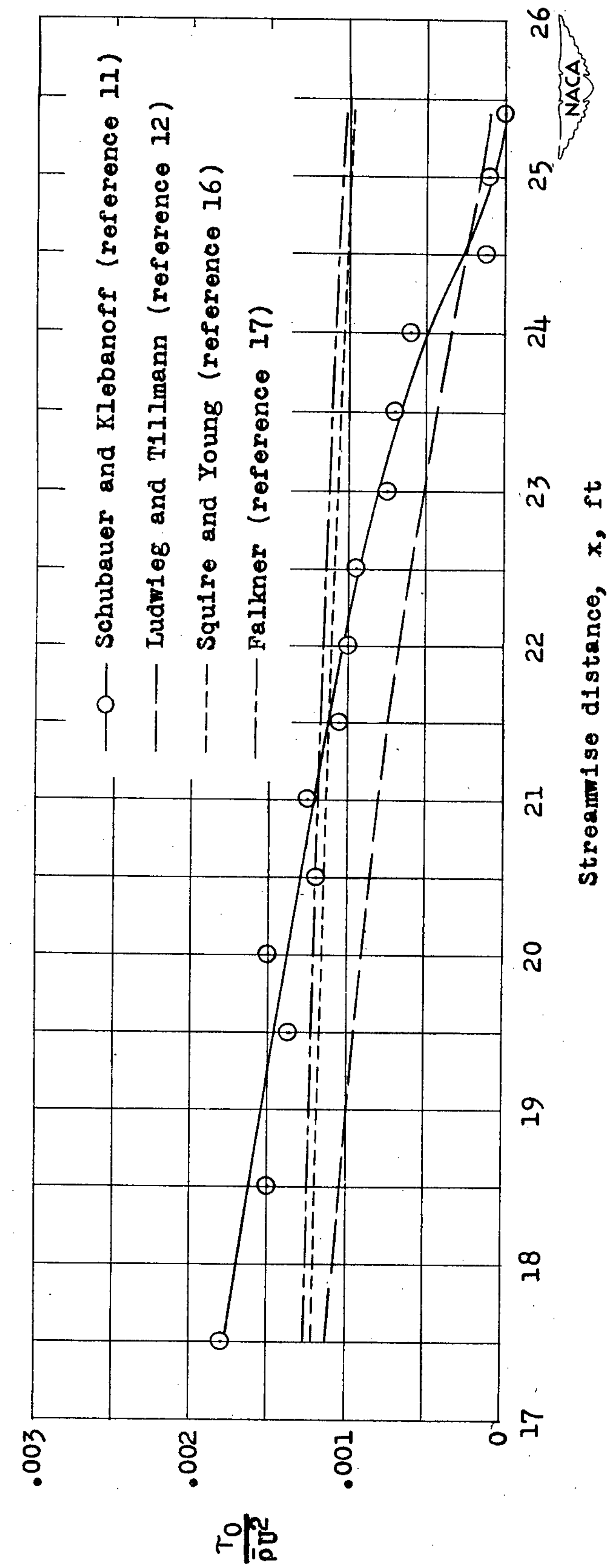

Figure 5.- Streamwise variation of the coefficient of wall shearing stress as obtained by various methods.

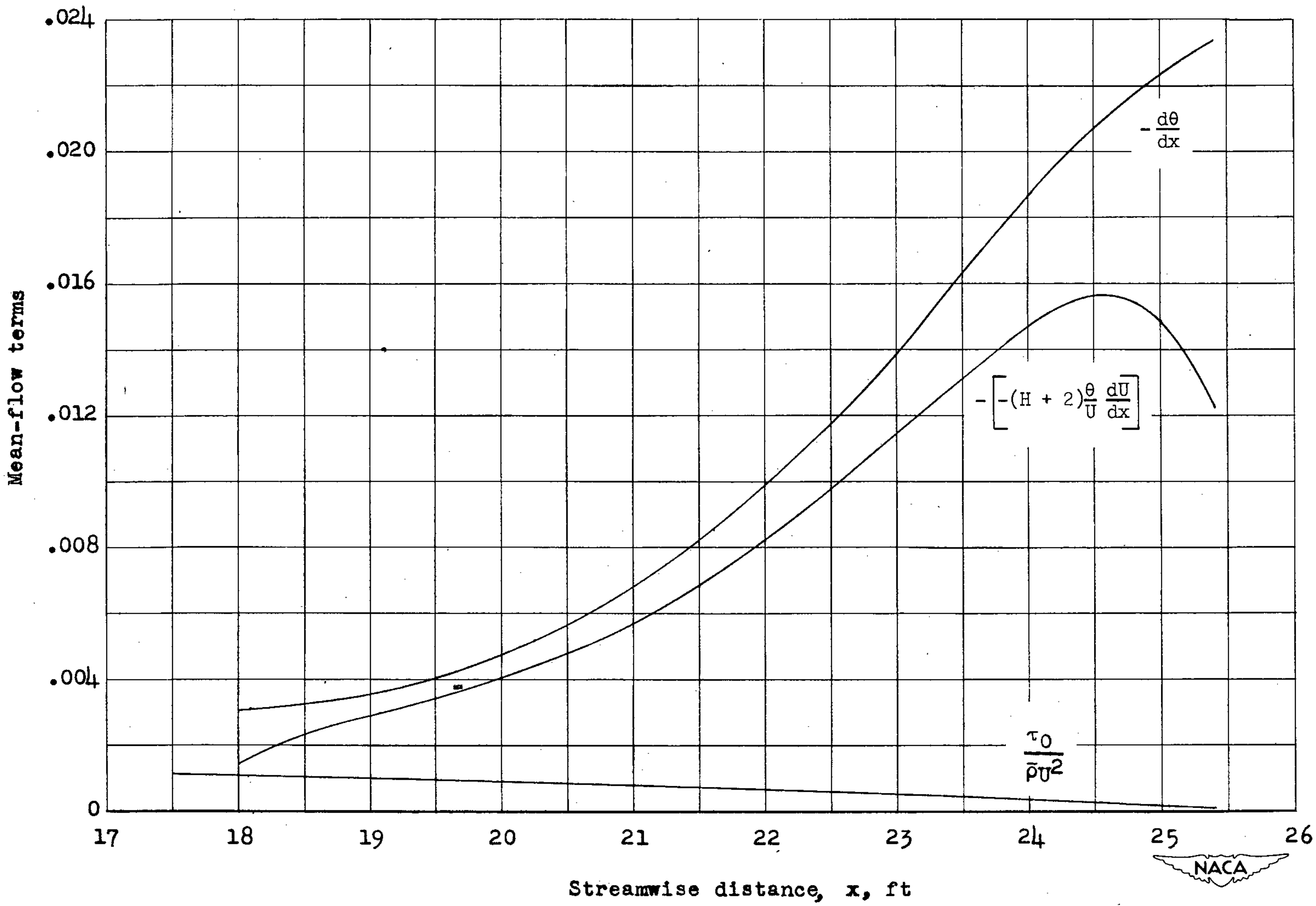

Figure 6.- Streamwise variation of the mean-flow terms in the Von Kármán momentum theorem as computed from the data of reference 11.

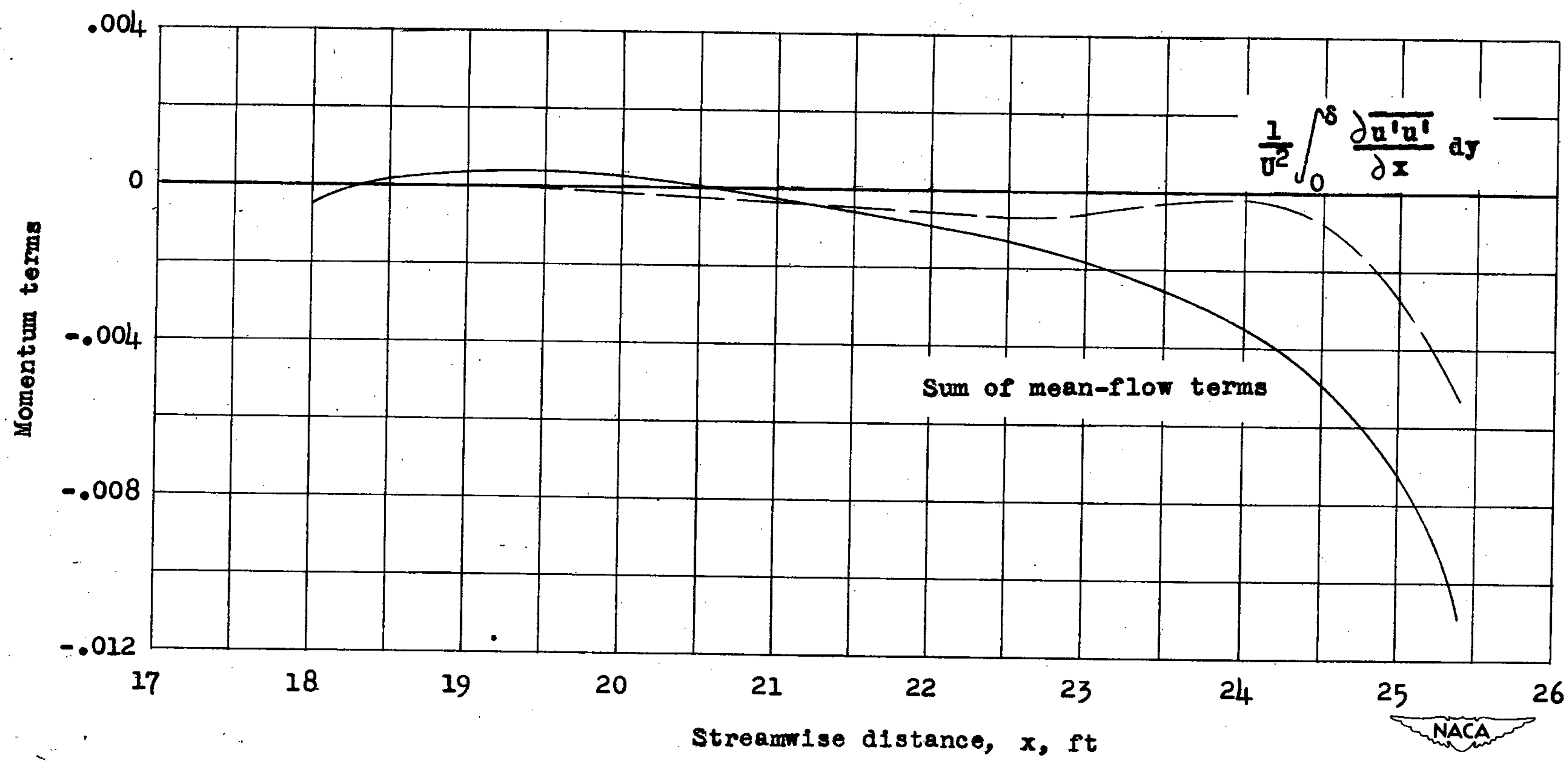

Figure 7.- Streamwise variation of the sum of the mean-flow terms in the Von Kármán momentum theorem; also the momentum term due to the turbulent longitudinal fluctuation.

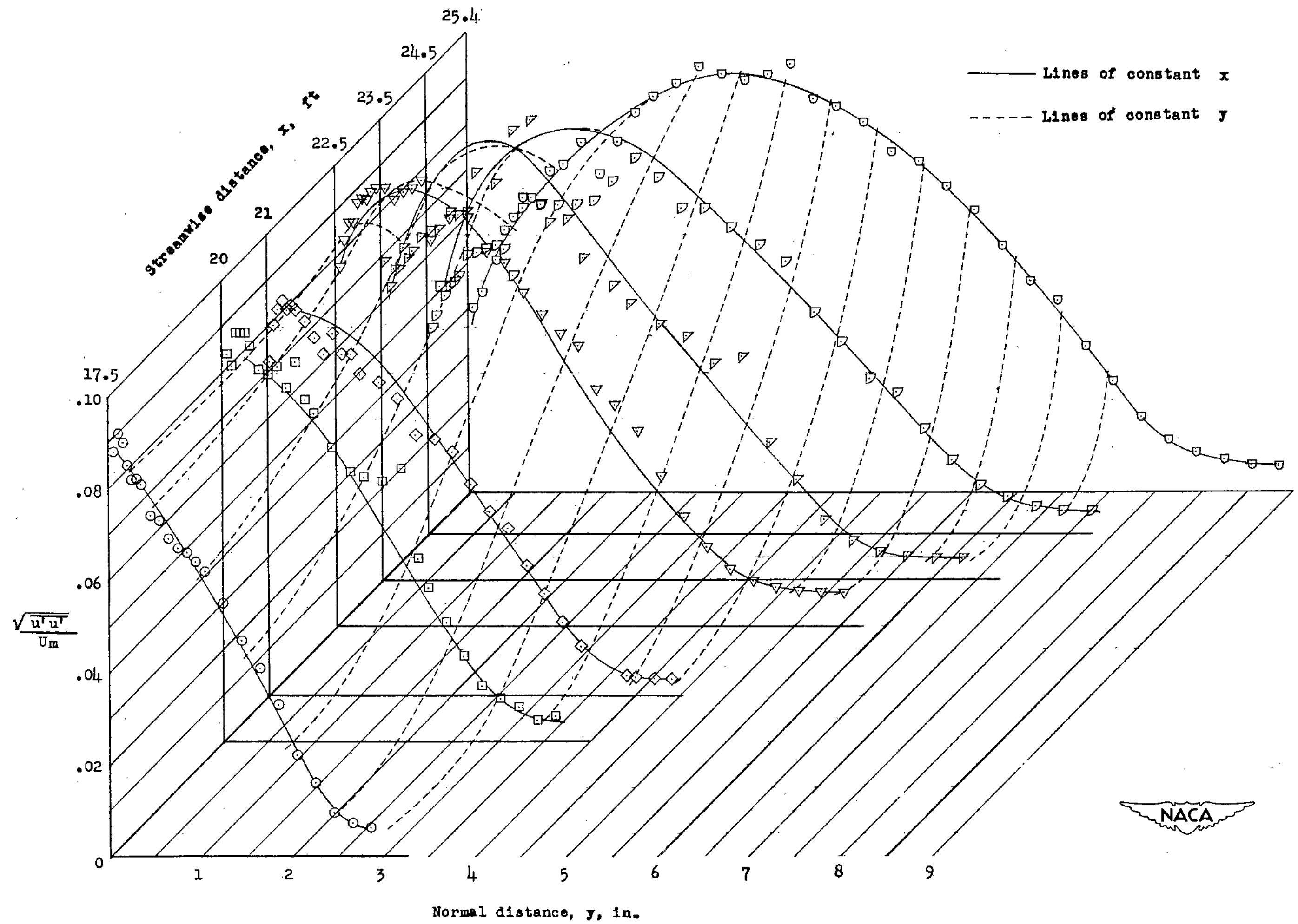

Figure 8.- Streamwise and normal variation of the turbulent longitudinal fluctuation ratio $\frac{\sqrt{\overline{u'u'}}}{U_m}$. Data were obtained from reference 11.